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p. 20

Low-Complexity Wavelet Filter Design for Image Compression

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Image compression algorithms based on the wavelet transform are an increasingly attractive and flexible alternative to other algorithms based on block orthogonal transforms. While the design of orthogonal wavelet filters has been studied in significant depth, the design of nonorthogonal wavelet filters, such as linear-phase (LP) filters, has not yet reached that point. Of particular interest are wavelet transforms with low complexity at the encoder. In this article, we present known and new parametrizations of two families of LP perfect reconstruction (PR) filters. The first family is that of all PR LP filters with finite impulse response (FIR), with equal complexity at the encoder and decoder. The second family is one of LP PR filters, which are FIR at the encoder and infinite impulse response (IIR) at the decoder, i.e., with controllable encoder complexity. These parametrizations are used to optimize the subband/wavelet transform coding gain, as defined for nonorthogonal wavelet transforms. Optimal LP wavelet filters are given for low levels of encoder complexity, as well as their corresponding integer approximations, to allow for applications limited to using integer arithmetic. These optimal LP filters yield larger coding gains than orthogonal filters with an equivalent complexity. The parametrizations described in this article can be used for the optimization of any other appropriate objective function.

I. Introduction

Tree-structured subband coding is an increasingly attractive and flexible alternative to other subband coding techniques based on block orthogonal transforms, which exhibit annoying blocky artifacts at low bit rates. The main building block of tree-structured subband coders is the two-channel subband coder. Given a number of levels of decomposition, L, the corresponding subband transform is a function of the analysis lowpass and highpass filters, $H_0(z)$ and $H_1(z)$, and the synthesis lowpass and highpass filters, $G_0(z)$ and $G_1(z)$ (see Fig. 1). Most filter banks of interest to subband coding are known as perfect reconstruction (PR) filter banks, and most filter banks with near-perfect reconstruction are approximations or truncations of PR filter banks.

Of particular interest to onboard image compression is the issue of computational complexity of the subband/wavelet transform; onboard resources are often limited, and so time and space complexity constraints are common. Also of particular interest are good subband transforms that are less complex than their inverse transforms, i.e., impose less complexity at the encoder, at the cost of more at the decoder. Since this feature cannot be obtained with the use of orthogonal transforms (which always have equal complexity at the encoder and at the decoder), most of our design efforts will be turned toward nonorthogonal filter banks, more precisely linear-phase (LP) filter banks.

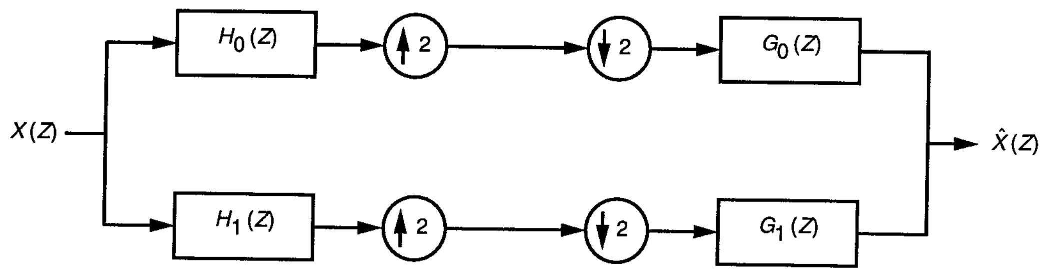

Fig. 1. Analysis and synthesis sections for a maximally decimated two-band filter bank.

In this article, we report on tree-structured subband coder (or wavelet filter) designs that satisfy complexity constraints at the encoder. First, we review the theory of PR filter banks and describe two families of solutions that are appropriate for our purposes. Then, we review definitions and known upper bounds on the subband coding gain of arbitrary linear transforms (not necessarily orthogonal). Finally, we derive solutions that maximize the subband coding gain, an incomplete, but informative measure of the performance of a subband transform-based coder, and give good integer approximations of these solutions. We conclude with the design of optimal boundary filters.

II. Two-Channel PR Filter Banks: A Review

Consider the two-channel filter bank presented in Fig. 1 (see [1] as a general reference).

To derive the equations satisfied by such a filter bank, an expression is needed for the z-transform of the reconstructed signal, $\hat{X}(z)$, in terms of the original signal, $X(z)$, and the analysis and synthesis filters, $H_k(z)$ and $G_k(z)$, respectively. The expressions for the $X_k(z)$ are given by

$$X_k(z) = H_k(z)X(z), \quad k = 0, 1 \tag{1}$$

After decimation, the z-transforms are given by

$$V_k(z) = \frac{1}{2}\left[X_k\left(z^{1/2}\right) + X_k\left(-z^{1/2}\right)\right], \quad k = 0, 1 \tag{2}$$

The z-transform of the signals $y_k(n)$ are given by

$$\begin{aligned} Y_k(z) &= V_k(z^2), \quad k = 0, 1 \\ &= \frac{1}{2}\left[X_k(z) + X_k(-z)\right] \\ &= \frac{1}{2}\left[H_k(z)X(z) + H_k(-z)X(-z)\right] \end{aligned} \tag{3}$$

Finally, the reconstructed signal becomes

$$\hat{X}(z) = \frac{1}{2}\left[G_0(z) G_1(z)\right] \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} \begin{bmatrix} X(z) \\ X(-z) \end{bmatrix} \tag{4}$$

$$\begin{aligned} \hat{X}(z) &= \frac{1}{2}\left(H_0(z)G_0(z) + H_1(z)G_1(z)\right)(X(z)) \\ &\quad + \frac{1}{2}\left(H_0(-z)G_0(z) + H_1(-z)G_1(z)\right)(X(-z)) \end{aligned} \tag{5}$$

For a PR filter bank, we must have, by definition,

$$\hat{X}(z) = z^{-l}(X(z))$$

i.e., the reconstructed signal $\hat{X}(z)$ is a delayed copy of input signal $X(z)$, which yields the following two PR conditions:

$$\begin{aligned} H_0(z)G_0(z) + H_1(z)G_1(z) &= 2z^{-l} \\ H_0(-z)G_0(z) + H_1(-z)G_1(z) &= 0 \end{aligned} \tag{6}$$

These equations can be rewritten as a linear system of two equations where $G_0(z)$ and $G_1(z)$ are the variables and $H_0(z)$ and $H_1(z)$ are assumed to be known:

$$\begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix} \begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \begin{bmatrix} 2z^{-l} \\ 0 \end{bmatrix} \tag{7}$$

Define the transfer function $Q(z)$ as

$$Q(z) = H_0(z)H_1(-z)$$

and define $\Delta(z)$ as

$$\Delta(z) = Q(z) - Q(-z)$$

Note that $\Delta(-z) = -\Delta(z)$. A well-known condition for a unique solution to exist is that the determinant of the matrix on the left side of Eq. (7) be nonzero. This yields the following condition on $Q(z)$:

$$\Delta(z) = Q(z) - Q(-z) \neq 0$$

which yields the following solution for $G_0(z)$ and $G_1(z)$:

$$\begin{aligned} G_0(z) &= 2z^{-l}\frac{H_1(-z)}{\Delta(z)} \\ G_1(z) &= -2z^{-l}\frac{H_0(-z)}{\Delta(z)} \end{aligned} \tag{8}$$

For finite impulse response (FIR) solutions to Eq. (8), one must require that $\Delta(z)$ be a simple delay ($\Delta(z) = 2 \times z^{-l}$), and so the equations now become

$$\begin{aligned} G_0(z) &= H_1(-z) \\ G_1(z) &= -H_0(-z) \end{aligned} \tag{9}$$

If $\Delta(z)$ is not a simple delay, then the PR synthesis filters $G_0(z)$ and $G_1(z)$ are infinite impulse response (IIR), irrespective of the choice of $H_0(z)$ and $H_1(z)$, i.e., whether the analysis filters are FIR or not.

The main two properties sought in PR filter banks for image compression are orthogonality and phase linearity. The only FIR wavelet filter that satisfies both properties simultaneously is the Haar filter. The IIR "sinc" wavelet [see Eq. (13)] also satisfies both properties simultaneously.

We now describe two families of LP PR filter banks among which we will look for those filters with the largest coding gain under some complexity constraints.

III. Linear-Phase PR Filter Banks

There are only two different types of LP (symmetric) filters (whether they are FIR or IIR), depending on whether there are one or two "center" coefficients. Symmetric filters with one center coefficient (odd-length for FIR filters) are called whole-sample symmetric (WSS), while those with two center coefficients (even-length for FIR filters) are called half-sample symmetric (HSS).

Since we are interested in solutions with low complexity at the encoder, we will consider the following solutions to the PR equations:

(1) Both the analysis and synthesis filters are FIR

(2) The analysis filters are FIR, while the synthesis filters are IIR (in practice, optimized truncations of IIR solutions are used)

A. FIR/FIR Solutions

Given that Eq. (9) is satisfied, the PR equations [see Eq. (6)] become

$$\Delta(z) = Q(z) - Q(-z) = H_0(z)H_1(-z) - H_0(-z)H_1(z) = 2z^{-l} \tag{10}$$

where the sum of the lengths of $H_0(z)$ and $H_1(z)$ ($|h_0| + |h_1|$) can be shown to always be a multiple of 4 (see [7]), and $l = (|h_0| + |h_1|)/2 - 1$. Given a choice for $H_0(z)$, there are an infinite number of solutions for $H_1(z)$ satisfying Eq. (10). The parametrization of this infinite family varies with the nature of the symmetry of the prototype filter $H_0(z)$: either HSS or WSS filters.

1. WSS Solutions. The following theorem (as well as its proof) is from [7]; it parametrizes all filters $H_1'(z)$ that are complementary to a prototype filter $H_0(z)$.

Theorem 1: *If the lengths $|h_0|$ and $|h_1|$ of the two complementary filters $H_0(z)$ and $H_1(z)$ are odd and satisfy*

$$|h_0| = |h_1| + 2$$

and if $H_0(-1) = 0$, then all the highpass analysis filters $H_1'(z)$ complementary to $H_0(z)$ are of the form

$$H_1'(z) = z^{-2m} H_1(z) + E(z^2) H_0(z)$$

where

$$E(z^2) = \sum_{i=1}^{m} \alpha_i \left(z^{-2(i-1)} + z^{-2(2m-i)} \right)$$

The length of $H_1'(z)$ is clearly $|h_1'| = |h_1| + 4m$. It is sometimes appropriate to impose zeros at π for lowpass filters ($H_0(e^{-j\pi}) = H_0(-1) = 0$) or zeros at dc frequency for highpass filters ($H_1(e^{j\pi}) = H_1(1) = 0$). Since $E(1) = 2\sum_{i=1}^{m} \alpha_i$ is nonzero in general, $H_0(-1) = 0$ is the only way to ensure that $H_1'(-1) = H_1(-1) = \sqrt{2}$. Finally, the requirement that $H_1(1) = 0$ (which we will find important later) translates into a constraint on the coefficients α_i:

$$H_1'(1) = H_1(1) + E(1)H_0(1) = H_1(1) + 2H_0(1) \sum_{i=1}^{m} \alpha_i$$

Therefore, to ensure that $H_1'(z)$ has a zero at dc frequency, we must have

$$\sum_{i=1}^{m} \alpha_i = -\frac{H_1(1)}{2H_0(1)} \tag{11}$$

If $H_1(1) = 0$, then we must have $\sum_{i=1}^{m} \alpha_i = 0$.

FIR filter banks will be referred to as $|h_0|/|h_1|$ (remember that $|g_0| = |h_1|$ and $|g_1| = |h_0|$). In Table 1, we give an example of the construction of a 5/7 PR filter pair from a 5/3 PR filter pair ($h_0 = [-1,\ 3,\ 4,\ 3,\ -1]$ and $h_1 = [-1,\ 3,\ -1]$) using Theorem 1. The solution with a zero at dc frequency is found using Eq. (11) and yields

$$\alpha_1 = \frac{-1}{16}$$

and

$$h_1' = \frac{[1,\ -3,\ -19,\ 42,\ -19,\ -3,\ 1]}{16}$$

Table 1. Generating a 5/7 filter from a 5/3 filter.

Filter	1	z^{-1}	z^{-2}	z^{-3}	z^{-4}	z^{-5}	z^{-6}
$z^{-2}H_1(z)$			-1	3	-1		
$H_0(z)$	-1	3	4	3	-1		
$z^{-2}H_0(z)$			-1	3	4	3	-1
$(1+z^{-2})H_0(z)$	-1	3	3	6	3	3	-1
$H_1'(z)$	$-\alpha_1$	$3\alpha_1$	$-1+3\alpha_1$	$3+6\alpha_1$	$-1+3\alpha_1$	$3\alpha_1$	$-\alpha_1$

2. HSS Solutions. While the following theorem (the equivalent to Theorem 1, but for HSS solutions) is not given in [7], it can be obtained in the same way as for WSS filters. We now give it without proof.

Theorem 2: *If the lengths $|h_0|$ and $|h_1|$ of the two complementary filters $H_0(z)$ and $H_1(z)$ are even and equal, then all the synthesis filters $H_1'(z)$ complementary to $H_0(z)$ are of the form*

$$H_1'(z) = z^{-2m} H_1(z) + E(z^2) H_0(z)$$

where

$$E(z^2) = \sum_{i=1}^{m} \alpha_i \left(z^{-2(i-1)} - z^{-2(2m-i+1)} \right)$$

Note that for HSS filters, there is always a zero at π for lowpass filters and at zero frequency for highpass filters, and so the construction implied by Theorem 2 always yields $H'(1) = H_1(-1) = 0$.

In Table 2, we give an example of the construction of a 2/6 PR filter pair from a 2/2 PR filter pair ($h_0 = [1,\ 1]$ and $h_1 = [1,\ -1]$) using Theorem 2. If we choose $\alpha_1 = 1/8$, we obtain a filter which has the largest possible number of zeros at dc frequency for a 2/6 PR filter: $h_1' = [1,\ 1,\ -8,\ 8,\ -1,\ -1]$.

Table 2. Generating a 2/6 filter from a 2/2 filter.

Filter	1	z^{-1}	z^{-2}	z^{-3}	z^{-4}	z^{-5}
$H_1(z)$			1	-1		
$H_0(z)$	1	1				
$z^{-4}H_0(z)$					1	1
$\alpha_1(1 - z^{-4})H_0(z)$	α_1	α_1			$-\alpha_1$	$-\alpha_1$
$H_1'(z)$	α_1	α_1	1	-1	$-\alpha_1$	$-\alpha_1$

B. FIR/IIR Solutions

Another interesting set of solutions to the PR equations involves IIR filters. We have seen previously that if $H_0(z)$ is given, there exist an infinite number of highpass filters $H_1(z)$ which are complementary to it. However, if $H_1(z)$ is chosen conveniently to satisfy

$$|H_1\left(e^{jw}\right)| = |H_0\left(e^{-jw}\right)| \tag{12}$$

then the solution for the synthesis filters is unique and is defined by Eq. (8). Again, HSS and WSS filters must be treated separately.

1. WSS Solutions. For WSS filters, a good choice, from a coding perspective, for $H_1(z)$ that satisfies Eq. (12) is

$$H_1(z) = -z^{-1} H_0(-z)$$

This yields

$$Q(z) = z^{-1}H_0(z)^2$$

and

$$\Delta(z) = z^{-1}(H_0(z)^2 + H_0(-z)^2)$$

Replacing in Eq. (8), we obtain the following IIR solutions for the synthesis filters:

$$G_0(z) = 2z^{-1}\frac{z^{-l}H_0(z)}{H_0(z)^2 + H_0(-z)^2}$$

and

$$G_1(z) = (-1)^l z\,(G_0(-z))$$

We now consider two examples that will prove useful later.

Example 1: If we choose $H_0(z) = (1 + z^{-1})^2$, then

$$Q(z) = z^{-1}(1 + z^{-1})^4$$

and

$$\Delta(z) = 2z^{-1}(1 + 6z^{-2} + z^{-4})$$

This yields the IIR solution for $G_0(z)$:

$$G_0(z) = \frac{(1 + z^{-1})^2}{1 + 6z^{-2} + z^{-4}}$$

Note that the product filter defined as $P(z) = H_0(z)G_0(z)$ in this last example is the Butterworth filter of order two (see [8] for further insights on this remark), i.e.,

$$P(z) = H_0(z)G_0(z) = \frac{(1 + z^{-1})^4}{1 + 6z^{-2} + z^{-4}}$$

Example 2: If we choose

$$H_0(z) = -1 + 7z^{-2} + 12z^{-3} + 7z^{-4} - z^{-6}$$

then $\Delta(z)$ becomes

$$\Delta(z) = 2z^{-1}\left(1 - 14z^{-2} + 35z^{-4} + 244z^{-6} + 35z^{-8} - 14z^{-10} + z^{-12}\right)$$

The product filter $Q(z)$ becomes

$$Q(z) = \frac{\left(-1 + 7z^{-2} + 12z^{-3} + 7z^{-4} - z^{-6}\right)\left(1 + z^{-1}\right)^2}{1 - 14z^{-2} + 35z^{-4} + 244z^{-6} + 35z^{-8} - 14z^{-10} + z^{-12}}$$

2. HSS Solutions. For HSS filters, the choice we make is

$$H_1(z) = H_0(-z)$$

which yields

$$Q(z) = H_0(z)^2$$

and

$$\Delta(z) = H_0(z)^2 - H_0(-z)^2$$

The IIR solution $G_0(z)$ is, therefore,

$$G_0(z) = (-1)^{l+1} \frac{2z^{-l} H_0(-z)}{H_0(z)^2 - H_0(-z)^2}$$

with

$$G_1(z) = (-1)^{l+1} G_1(z)$$

We now consider an FIR inversion example that will prove useful later.

Example 3: If we choose

$$H_0(z) = -1 + 2z^{-1} + 9z^{-2} + 9z^{-3} + 2z^{-4} - z^{-5}$$

then $\Delta(z)$ becomes

$$\Delta(z) = z^{-1}\left(-8 + 36z^{-2} + 344z^{-4} + 36z^{-6} - 8z^{-8}\right)$$

and we obtain the following IIR solution for the synthesis lowpass filter:

$$G_0(z) = \frac{-1 + 2z^{-1} + 9z^{-2} + 9z^{-3} + 2z^{-4} - z^{-5}}{-8 + 36z^{-2} + 344z^{-4} + 36z^{-6} - 8z^{-8}}$$

IV. The Subband Coding Gain for PR Filter Banks

There exists a measure of the coding performance of transform-based coding schemes known as the transform coding gain [4]. The coding gain is simply defined as the ratio of the reconstruction error variance of the transform coding scheme and the reconstruction error variance of a pulse code modulation (PCM) scheme (i.e., without a linear transformation to decorrelate the signal). If the coding gain is expressed in decibels (dB), then a coding gain of 0 for a particular transform means that no gain is achieved by applying the transform to the signal. The expression most widely known is one that assumes very fine scalar quantization of the transformed coefficients and that is independent of the quantization level under that assumption. The coding gain then becomes a function of the linear transform only, and the source correlation model. For example, the coding gain of the 8-point discrete cosine transform is 8.83 dB, using a first-order Markov source model with correlation $\rho = 0.95$. Other correlation models can also be used, as well as the actual correlation statistics of the signal to be coded. The coding gain is, therefore, the proper measure of coding performance when high-rate quantization is assumed (i.e., at low compression ratios). At low bit rates, the approximations that led to its expression are no longer valid, and so the coding performance of the given transform is no longer guaranteed. It can be said, however, that quite often transforms that have a large coding gain at high bit rates tend to perform well at lower bit rates, as can be observed from actual rate-distortion curves [2].

In [3], Katto and Yasuda derive an expression for the coding gain that is valid for nonunitary subband transforms (such as the biorthogonal wavelet transform). If M is the number of subbands, $h_k(i)$ ($k = 1, \ldots, M$) are the coefficients of the kth analysis filter, $g_k(j)$ ($k = 1, \ldots, M$) are the coefficients of the kth synthesis filter, and ρ is the correlation of the source modeled as a one-dimensional Markov source, their expression for the coding gain is

$$G_{SBC}(\rho) = \frac{1}{\Pi_{k=1}^{M}(A_k B_k)^{\alpha_k}}$$

where

$$A_k = \sum_i \sum_j h_k(i) h_k(j) \rho^{|j-i|}$$

$$B_k = \sum_i g_k(i)^2$$

and α_k is the subsampling ratio for the kth filter (i.e., $\alpha_k = 1/M$ for a uniform subband decomposition). The coding gain for the two-dimensional case is simply twice the value in the one-dimensional case.

Note that, for tree-structured subband decompositions, G_{SBC} is a function of the following:

(1) The subband decomposition, which can be entirely defined by a binary tree

(2) The analysis lowpass and highpass filters, $H_0(z)$ and $H_1(z)$, and the synthesis lowpass and highpass filters, $G_0(z)$ and $G_1(z)$

(3) The source correlation model

We will consider two types of subband decomposition: "logarithmic" decompositions, for which only the downsampled output of the analysis lowpass filter may be further decomposed, and "arbitrary" decompositions, for which there is no constraint on which downsampled output may be decomposed next.

A "uniform" decomposition up to level L would correspond to a binary tree of depth L. The L will always be referred to as the maximum depth of the tree. The analysis lowpass and highpass filters, $H_0(z)$ and $H_1(z)$, and the synthesis lowpass and highpass filters, $G_0(z)$ and $G_1(z)$, are required to satisfy the PR conditions (Fig. 1). The model used is a two-dimensional separable Markov model. Even though our results will be limited to a Markov model, the design methodology used can easily be extended to any other source correlation model.

First, we review the known upper bounds on the subband coding gain. Then we examine the limiting case $\rho \to 1$ and show that a necessary condition for the coding gain to be adequate as $\rho \to 1$ is for $H_1(z)$ to have a zero at dc frequency. Finally, we describe a useful normalization of the coding gain, which compares coding gain values to the known upper bounds for orthogonal wavelet filters.

A. Known Upper Bounds

Almost all the theoretical results on subband transform coding are about orthogonal subband transforms. We know that the largest coding gain achievable by any linear transformation for a Markov source with correlation ρ is that obtained by the Karhunen–Loeve transform (KLT) (an orthogonal transform) as the block size goes to $+\infty$:

$$G_{KLT}(\rho) = \frac{1}{1-\rho^2}$$

In [5], de Queiroz and Malvar give an expression for the largest coding gain attainable by any orthogonal subband transform for depth-L logarithmic subband decompositions, which corresponds to the coding gain of the IIR sinc filter:

$$h_k = \frac{\sin(\pi k/2)}{\pi k/2}, \quad -\infty < k < +\infty \tag{13}$$

the frequency response of which is an ideal lowpass filter. They use it to show that orthogonal wavelet transforms are asymptotically suboptimal. We will refer to that expression as $G_{LT}(\rho, L)$, where L is the number of levels of decomposition of the transform.

The same derivation used in [5] can be used to obtain an expression for the largest coding gain attainable by any orthogonal subband transform for depth-L uniform subband decompositions; we note it as $G_{UT}(\rho, L)$. Naturally, in the limit as $L \to +\infty$, we have $G_{UT}(\rho, L) \to G_{KLT}(\rho)$. Therefore, the subband coding gain of any orthogonal subband transform of maximum depth L is upper bounded by $G_{LT}(\rho, L)$ for logarithmic decompositions and by $G_{UT}(\rho, L)$ for uniform decompositions.

B. Limiting Case: $\rho \to 1$

For values of ρ close to 1, only the expression for A_k changes when evaluating the coding gain $G_{SBC}(\rho)$. By writing $\rho = 1 - \epsilon$, and assuming $\epsilon \approx 0$, we obtain for A_k

$$A_k = \sum_i \sum_j h_k(i) h_k(j) (1-\epsilon)^{|j-i|}$$

$$A_k \approx \sum_i \sum_j h_k(i) h_k(j) (1 - |j-i|\epsilon)$$

$$A_k \approx \left(\sum_i h_k(i)\right)^2 - \left(\sum_i \sum_j h_k(i)h_k(j)|j-i|\right)\epsilon$$

If $\sum_i h_k(i) = 0$, the A_k simplifies to

$$A_k \approx -\left(\sum_i \sum_j h_k(i)h_k(j)|j-i|\right)\epsilon$$

Otherwise, it simplifies to

$$A_k \approx \left(\sum_i h_k(i)\right)^2$$

Therefore, in the limit as $\epsilon \to 0$, the coding gain is significantly larger by imposing $\sum_i h_k(i) = 0$, i.e., $H_1(1) = 0$, corresponding to a zero at π.

C. Coding Gain Notations

In the following, only dB values of the normalized coding gain (NCG) will be given. The NCG is obtained by computing the dB value of the subband coding gain, and subtracting the dB value of $G_{LT}(\rho, L)$, so that the wide range of coding values can be displayed on the same graph. All NCG values will be given for values of ρ between 1/2 and 1. As mentioned earlier, logarithmic and arbitrary subband decompositions will be considered. For logarithmic decompositions, values of the NCG will be given for a maximum depth L equal to 8, while for arbitrary decompositions, we chose $L = 6$. This means that the NCG values will slightly vary for logarithmic and arbitrary decompositions, although they remain comparable, due to the limited changes in subband coding gain for $L = 6$ and $L = 8$.

In all NCG curves (Fig. 2), the highest solid curve (which we refer to as the reference UT curve) corresponds to the upper bound provided by the $G_{UT}(\rho, L)$, itself normalized by subtracting the dB value of $G_{LT}(\rho, L)$. The other solid curve (flat and equal to 0) is simply the reference curve for logarithmic decompositions (which we refer to as the reference LT curve), i.e., the dB value of $G_{LT}(\rho, L)$. The gap between the two solid curves is simply the coding gain gap mentioned in [5] between the optimal uniform and logarithmic subband decompositions for orthogonal filters.

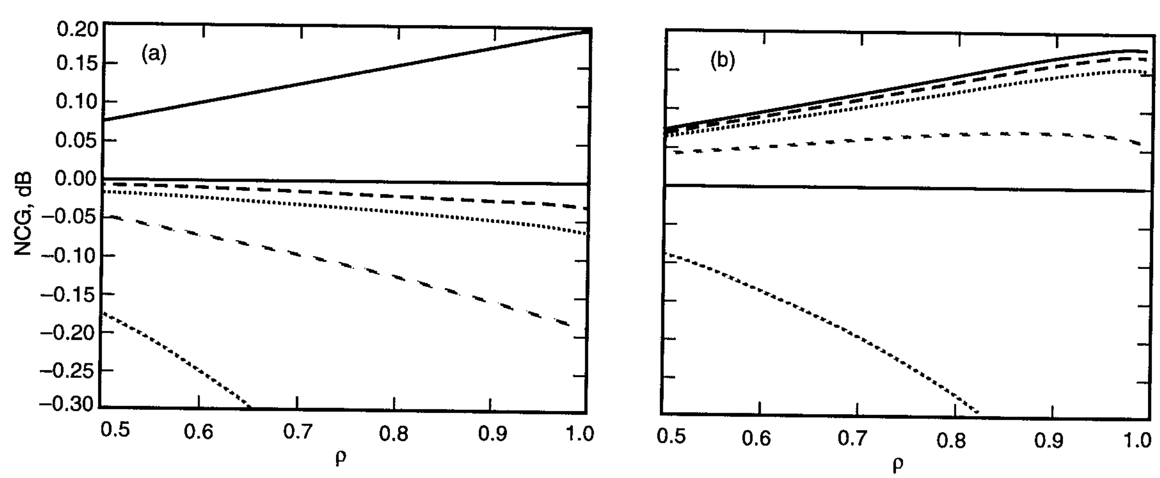

Fig. 2. Normalized coding gain of optimized n-tap orthogonal filters (n = 4, 8, 14, 20): (a) 8-level logarithmic decomposition orthogonal filter and (b) 6-level uniform decomposition orthogonal filter.

V. Optimal PR Filters

Given the expression for the subband coding gain, and a parametrization of HSS and WSS filters, we now look for filters with optimal coding gain performance as a function of average filter length. Then, we will look for integer approximations of these filters which yield similar coding gain performance. We now briefly review known results on orthogonal filters which maximize the subband coding gain [9].

A. Orthogonal Filters

The parametrization used here is the one described in [6], a parametrization of even n-tap orthogonal filters with one zero at π using $n/2-1$ free parameters. The optimization was performed for logarithmic and uniform decompositions for five levels of decomposition.

The NCG curves are provided in Fig. 2 for the usual values of $n = 4,\ 8,\ 14$, and 20, and were computed for 8 and 6 levels of decomposition, even though the optimization was done for $L = 5$. The corresponding coding gains for $L = 5$ and $\rho = 0.95$ can be found in Table 3. The filters obtained were very similar for both types of decomposition, and similar to those found in [9]. The filters obtained are local maxima of the coding gain, and are believed to be very close to the global maxima. An interesting characteristic of these filters is that they have a maximum number of roots on the unit circle.

Table 3. Coding gain performance of optimal FIR/FIR filters, logarithmic decompositions, $L = 5$, $\rho = 0.95$, for orthogonal filters.

Filter length	Coding gain, dB
2	8.24
4	9.29
6	9.62
8	9.74
14	9.85
∞	9.91

B. LP Biorthogonal Filters

Optimization of the subband coding gain is possible given an efficient parametrization of the various families of filters. The parametrization of FIR/FIR solutions is more complex due to the many combinations of filter lengths for a given average filter length. The parametrization of FIR/IIR solutions is simpler, since the parametrization of $H_0(z)$ is all that is required.

1. FIR/FIR Solutions. Given an even average filter length n, the possible filter length combinations $|h_0|/|h_1|$ are of the type $|h_0| = k$ and $|h_1| = n-k$, with $k = 1, \ldots, n/2$. This means that the filters can be either HSS or WSS, resulting in different parametrizations.

For WSS filters, Theorem 1 describes how to generate all the filters $H_1'(z)$ of length $|h_0| - 2 + 4m$ complementary to a filter $H_0(z)$. In [7], it is shown that there exists a unique filter $H_1(z)$ complementary to $H_0(z)$ with $|h_1| = |h_0| - 2$. A useful consequence of that result is that there exists a unique filter $H_1(z)$ complementary to $H_0(z)$ with $|h_1| = |h_0| + 2$ with the added constraint that $H_1(1) = 0$. These unique solutions can easily be found by solving a system of linear equations corresponding to the PR conditions. Parametrizing WSS filters is, therefore, done in two steps: (1) parametrizing a filter $H_0(z)$ and calculating its unique complementary filter $H_1(z)$ (with $|h_1| = |h_0| \pm 2$ depending on whether $H_1(1) = 0$ is required or not) and (2) parametrizing all the filters $H_1'(z)$ of higher order and complementary to $H_0(z)$. Note that if the desired lengths $|h_0|$ and $|h_1|$ satisfy $|h_0| < |h_1|$, one simply can replace the analysis filters by the synthesis filters and note that $|g_1| = |h_0| < |h_1| = |g_0|$, allowing the suggested parametrization.

For HSS filters, Theorem 2 describes how to generate all the filters $H_1'(z)$ of length $|h_0| + 4m$ complementary to a filter $H_0(z)$. The parametrization again takes place in two steps: (1) parametrizing a filter $H_0(z)$ and calculating its unique complementary filter $H_1(z)$ (with $|h_1| = |h_0|$) and (2) parametrizing all the filters $H_1'(z)$ of higher order and complementary to $H_0(z)$. For the case of $|h_0| < |h_1|$, the same remark made above concerning WSS filters applies here.

All optimizations were carried out for average filter lengths of $n = 4$, 6, and 8, with $\rho = 0.95$ and $L = 5$ levels of logarithmic decomposition. For $n = 14$, only the 17/11 filter length combination was examined. $H_1(1) = 0$ is imposed on all solutions. The optimal filter coefficients are given in Table 4 and the corresponding coding gain values in Table 5. In Table 5, note the higher coding gains obtained by LP filters, even higher than the upper bound for orthogonal filters (Table 3)!

Table 4. Optimal linear-phase FIR/FIR filter banks.

i	$(h_0)_{\pm i}$	$(h_1)_{\pm i}$	i	$(h_0)_i, (h_0)_{-1-i}$	$(h_1)_i, (h_1)_{-1-i}$
	5/3 Filter			2/6 Filter	
0	1.02707904	0.70710678	1	0.70710678	0.70710678
1	0.38713452	−0.35355339	2		0.09733489
2	−0.19356726		3		−0.09733489
	5/7 Filter			6/10 Filter	
0	0.95902785	0.75833803	1	0.79363797	0.60082030
1	0.36569130	−0.36322679	2	0.08023230	−0.14072753
2	−0.13809844	−0.02561563	3	−0.16676349	−0.07121045
3		0.00967340	4		−0.01194390
			5		0.02482550
	9/7 Filter				
0	0.81096744	0.79365640			
1	0.39424588	−0.43412065			
2	−0.11475353	−0.04327481			
3	−0.02568087	0.08056725			
4	0.04781158				
	17/11 Filter				
0	0.83851308	0.70235757			
1	0.45656233	−0.41589851			
2	−0.09573748	−0.02337038			
3	−0.11802962	0.09492166			
4	0.06386749	0.02574498			
5	0.01728699	−0.03257654			
6	−0.03776016				
7	−0.00625838				
8	0.00791907				

Table 5. Coding gain performance of optimal FIR/FIR filters, logarithmic decompositions, $L = 5$, $\rho = 0.95$, for LP filters.

Mean filter length	Coding gain	$\|h_0\|/\|h_1\|$
2	8.24	2/2
4	9.60	5/3
6	9.71	5/7
8	9.88	9/7
14	9.96	17/11
∞	?	?

2. FIR/IIR Solutions. For FIR/IIR solutions to the PR equations, the parametrization is limited to that of the prototype filter $H_0(z)$. The corresponding inverse IIR filters are unique and can be approximated by long enough truncations of the IIR filters for coding gain calculations.

All optimizations were carried out for $L = 5$ levels of logarithmic decompositions and $\rho = 0.95$. $H_1(1) = 0$ (equivalently $H_0(-1) = 0$) is imposed on all solutions. Filter lengths considered for $H_0(z)$ (and therefore $H_1(z)$) were $n = 3, \ldots, 11$. The optimal coding gains obtained are given in Table 6 as a function of filter length. The only filter lengths of interest from a coding gain perspective are clearly $n = 3$, 6, and 7; larger coding gains are attainable with FIR/FIR solutions for $n \geq 8$. The corresponding optimal analysis filter coefficients for $n = 3$, 6, and 7 are given in Table 7. Their IIR inverses can be computed using Eq. (8).

Table 6. Optimal coding gain as a function of filter length for FIR/IIR combinations.

Filter length	Coding gain, dB for $\rho = 0.95$
3	9.36
4	9.33
5	9.36
6	9.70
7	9.76
8	9.74
9	9.79
10	9.84
11	9.84

3. Integer Approximations of Optimal LP Filters. Because of the complexity constraints imposed by onboard processing, it is often useful to look for integer approximations of good filters. Rather than conduct an exhaustive search of all PR solutions with integer coefficients, we narrowed our search to a neighborhood of the optimal filters arrived at earlier. We also restricted the coding gain of the integer approximation to be within 0.05 dB of the coding gain of the filter it is approximating, a constraint that always leads to solutions of minimum integer ranges.

Good integer approximations to optimal FIR/FIR filters are given in Table 8. Again, $H_1(1) = 0$ is required of all integer solutions. A few others exist and are in the immediate neighborhood of those given.

Table 7. Optimal analysis FIR filters with inverse IIR filters.

Length 3	
i	$(h_0)_{\pm i}$
0	0.70710678
1	0.35355339

Length 6	
i	$(h_0)_i, (h_0)_{-1-i}$
1	0.64546192
2	0.13959559
3	−0.07795073

Length 7	
i	$(h_0)_{\pm i}$
0	0.75048476
1	0.42976974
2	−0.02168899
3	−0.07621635

Table 8. Integer approximations of optimal analysis FIR/FIR filters; the coding gain (CG) is given for $\rho = 0.95$ and $L = 5$.

$\lvert h_0\rvert/\lvert h_1\rvert$	h_0	h_1	CG
2/6	[1, 1]	[1, 1, −8, 8, −1, −1]	9.59
5/3	[−1, 2, 6, 2, −1]	[−1, 2, −1]	9.59
6/6	[−1, −2, 32, 32, −2, −1]	[3, 6, −32, 32, −6, −3]	9.68
5/7	[−1, 3, 8, 3, −1]	[1, −3, −31, 66, −31, −3, 1]	9.70
9/7	[2, −1, −6, 19, 44, 19, −6, −1, 2]	[2, −1, −12, 22, −12, −1, 2]	9.86
6/10	[−2, 1, 10, 10, 1, −2]	[−2, 1, 6, 12, −57, 57, −12, −6, −1, 2]	9.87

Good integer approximations to the analysis FIR filters of FIR/IIR solutions are given in Table 9. Again, $H_1(1) = 0$ is required on all integer solutions. The integer range is, not surprisingly, lower than that of the integer FIR/FIR solutions, since the complexity (here the integer range) is minimized at the encoder, at the cost of more complexity at the decoder. The truncated IIR inverses of these three filters can be found in Table 10, while their closed-form expressions were derived earlier in Examples 1–3 (Section III.B). Actual implementation of these inverse filters should involve truncations of the IIR inverses, so as to minimize the average or maximum reconstruction error, for example.

Table 9. Integer approximations of optimal analysis FIR filters with inverse IIR filters; the CG is given for $\rho = 0.95$ and $L = 5$.

Filter length	Filter coefficients	Coding gain
3	[1, 2, 1]	9.36
6	[−1, 2, 9, 9, 2, −1]	9.69
7	[−1, 0, 7, 12, 7, 0, −1]	9.74

Table 10. IIR inverses of FIR filters with integer coefficients: coefficients with 8 decimal points only.

	$h_0 = [1,\ 2,\ 1]$	$h_0 = [-1,\ 2,\ 9,\ 9,\ 2,\ -1]$	$h_0 = [-1,\ 0,\ 7,\ 12,\ 7,\ 0,\ -1]$
i	$(g_0)_{\pm i}$	$(g_0)_i,\ (g_0)_{-1-i}$	$(g_0)_{\pm i}$
0	1.00000000	0.74953169	0.88621564
1	0.41421356	0.08132065	0.43634598
2	−0.17157288	−0.16264131	−0.14842319
3	−0.07106781	0.00882291	−0.11356570
4	0.02943725	0.03529163	0.07681856
5	0.01219331	0.00095724	0.04203769
6	−0.00505063	−0.00765795	−0.02465968
7	−0.00209204	0.00010386	−0.01521935
8	0.00086655	0.00166170	0.00908314
9	0.00035894	0.00001127	0.00537790
10	−0.00014868	−0.00036057	−0.00322211
11	−0.00006158	0.00000122	−0.00193014
12	0.00002551	0.00007824	0.00115210
13	0.00001057	0.00000013	0.00068826
14	−0.00000438	−0.00001698	−0.00041153
15	−0.00000181	0.00000001	−0.00024596
16	0.00000075	0.00000368	0.00014697
17	0.00000031	0.00000000	0.00008784
18	−0.00000013	−0.00000080	−0.00005250
19	−0.00000005	0.00000000	−0.00003138
20	0.00000002	0.00000017	0.00001875
21	0.00000001	0.00000000	0.00001121
22		−0.00000004	−0.00000670
23		0.00000000	−0.00000400
24		0.00000001	0.00000239
25			0.00000143
26			−0.00000085
27			−0.00000051
28			0.00000031
29			0.00000018
30			−0.00000011
31			−0.00000007
32			0.00000004
33			0.00000002
34			−0.00000001
35			−0.00000001

VI. Optimal Boundary Filters

The standard signal extension technique used for LP filters is known as the symmetric extension technique. We propose to choose the signal extension that will yield the largest coding gain and to compare the coding gain thus attained with that obtained by both the symmetric and circular extension techniques. Since there is an equivalence between signal extension and modification of the filters at the left and right boundaries of the signal, we will now talk exclusively about boundary filter design.

We will illustrate our method with the optimization of the boundary filters for the 5/3 filter:

$$H_0(z) = [-1,\ 2,\ 6,\ 2,\ -1]$$

and

$$H_1(z) = [-1,\ 2,\ -1]$$

Consider a signal of length 8. The following matrix A,

$$A = \begin{pmatrix} a & b & c & & & & \\ -2 & 4 & -2 & & & & \\ -1 & 2 & 6 & 2 & -1 & & \\ & & & \ddots & & & \\ & & & -2 & 4 & -2 & \\ & & & d & e & f & g \\ & & & & & h & i \end{pmatrix}$$

corresponds to the parametrization of the linear transform that is equivalent to the one-level decomposition of a signal of length 8, with the parameters a, b, c, d, e, f, g, h, and i. The inverse matrix B should be of the type

$$B = 32 \times A^{-1} = \begin{pmatrix} ? & ? & & & & & & \\ ? & ? & 2 & -1 & & & & \\ & ? & 4 & 2 & & & & \\ & ? & 2 & 6 & & -1 & & \\ & & & 2 & \ddots & -2 & & \\ & & & -1 & & ? & ? & ? \\ & & & & & ? & ? & ? \\ & & & & & ? & ? & ? \end{pmatrix}$$

where the question marks indicate unspecified values that depend on the choice of the parameters of matrix A. A simple analysis shows that biorthogonality is preserved if the following equations are satisfied:

$$a + b + c = 8$$

$$b + 2c = 0$$

$$2d + e = 0$$

$$
\begin{aligned}
d+e+f+g &= 8 \\
h+i &= 0 \\
-h+i &= 8
\end{aligned}
\tag{14}
$$

which yields the parameters as a function of a, f, and g only.

$$
\begin{aligned}
c &= a-8 \\
b &= -2c = -2a+16 \\
d &= f+g-8 \\
e &= -2d = -2f-2g+16 \\
h &= -4 \\
i &= 4
\end{aligned}
\tag{15}
$$

We limited our search to integer values of the three parameters, a, f, and g, and obtained the largest coding gain for a five-level decomposition with the following parameter values: $a=6$, $f=3$, and $g=4$, i.e., $b=4$, $c=-2$, $d=-1$, $e=2$, $h=-4$, and $i=4$.

$$
A = \begin{pmatrix}
6 & 4 & -2 & & & & \\
-2 & 4 & -2 & & & & \\
-1 & 2 & 6 & 2 & -1 & & \\
 & & & \ddots & & & \\
 & & & -2 & 4 & -2 & \\
 & & & -1 & 2 & 3 & 4 \\
 & & & & & -4 & 4
\end{pmatrix}
$$

and

$$
B = \begin{pmatrix}
4 & -4 & & & & & & \\
2 & 5 & 2 & -1 & & & & \\
 & -2 & 4 & 2 & & & & \\
 & -1 & 2 & 6 & & -1 & & \\
 & & & 2 & \ddots & -2 & & \\
 & & & -1 & & 6 & 2 & -2 \\
 & & & & & -2 & 4 & -4 \\
 & & & & & -2 & 4 & 4
\end{pmatrix}
$$

In Table 11, we give the coding gain that was computed for five levels of logarithmic decomposition, with $\rho = 0.95$, for the three types of signal extension and various signal lengths. The difference between symmetric and optimal extension becomes insignificant at large signal lengths, but is important for smaller ones, and so the use of optimal extensions is relevant, particularly in applications in which an image might be divided into smaller separate blocks for compression.

Table 11. Coding gain performance of finite length signals with 5/3 filter.

Signal length	Circular	Symmetric	Optimal
64	9.16	9.28	9.39
128	9.37	9.44	9.49
256	9.48	9.51	9.54
512	9.55	9.55	9.56
∞	9.59	9.59	9.59

VII. Conclusion

Two families of LP filter banks were examined for efficient implementation as part of an onboard image compression system. The first family (FIR/FIR solutions to the PR equations) exhibits the same complexity at the encoder and the decoder. The shortest filters of that family with the best coding gain performance were designed using a parametrization scheme specific to that family. Integer approximations for that family of filters with similar coding gain performance were also designed. The coding gain performance of this family of LP filters is better than that of orthogonal filters for logarithmic decompositions; for long enough filters, their coding gain is above the upper bound over all possible orthogonal filters. The second family (FIR/IIR solutions) exhibits an asymmetry between the computational complexity at the encoder and the decoder, allowing for less complexity at the encoder, at the cost of more complexity at the decoder. Filters that maximize the subband coding gain were designed, as well as integer approximations with similar coding gain performance. The range of the coefficients of the integer approximations was found to be less than that of counterparts from the first family of LP filters with equivalent performance, validating the approach consisting of designing filters with asymmetric characteristics at the encoder and at the decoder.

These summarized results point to the following conclusions when comparing orthogonal and biorthogonal wavelet filters:

(1) For the same average filter length, biorthogonal wavelet filters yield larger coding gains than their orthogonal counterparts, when logarithmic decompositions are considered

(2) Biorthogonal wavelet filters can be used in asymmetric applications, which require very low complexity at the encoder (or the decoder), unlike orthogonal filters

Further investigations that incorporate other design criteria, particularly at low bit rates such as minimal ringing around edges (usually obtained with short filters) and a visual evaluation of the distortion present in the reconstructed images, are required in wavelet filter design. This study illustrates the improved design flexibility of biorthogonal wavelet filters over orthogonal filters, which is likely to be confirmed when using objective functions other than the coding gain.

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